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# Linear Stochastic Differential Equations Driven by a Fractional Brownian Motion with Hurst Parameter less than $1/2$

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**Abstract**

In this paper we use the chaos decomposition approach to establish the existence of a unique continuous solution to linear fractional differential equations of the Skorohod type. Here the coefficients are deterministic, the inital condition is anticipating and the underlying fractional Brownian motion has Hurst parameter less than 1/2. We provide an explicit expression for the chaos decomposition of the solution in order to show our results.

## 1 Introduction

The fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is a Gaussian process with useful properties. In particular, the stationary of its increments, and the self–similarity and the long–range dependence of this process (see Mandelbrot and Van Ness [13]) become the fBm a suitable driven noise for the construction of stochastic models and the analysis of phenomena that exhibit scale–invariant and long–range correlated force. However the fBm is not a semimartingale when $H \neq \frac{1}{2}$. Hence we cannot apply the techniques of the stochastic calculus in the Itô sense to define a stochastic integral with respect to the fBm.

Different interpretations of stochastic integral with respect to the fBm $B$ have been used by several authors to study the fractional stochastic differential equation of the form

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s, \quad 0 \le t \le T. \tag{1.1}$$

In the case that $H \in (1/2, 1)$, it is reasonable to consider equation (1.1) as a path–by–path ordinary differential equation since $B$ has Hölder–continuous paths with all exponents less than $H$ and $\int_0^T Y_s dB_s$ exists as a pathwise Riemann–Stieltjes integral for any $\lambda$–Hölder continuous process $Y$ with $\lambda > 1 - H$ (see Young [22]). This pathwise equation has been studied by several authors (see for instance [7, 8, 11, 19] and [23]). Zähle [23] has improved this pathwise approach

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by constructing and extension of the Lebesgue–Stieltjes integral via the fractional calculus [20]. These extended integral has been considered by Nualart and Răşcanu [16] and Zähle [24] to analyze equation (1.1).

The notion of $p$–variation and a limit result of Lyons [12] allow Coutin and Qian [2, 3] to get a Wong–Zakai type approximation limit for equation (1.1) when $H > 1/4$.

Alòs et al [1] (resp. León and Tudor [10]) work with the stochastic Stratonovich integral in the Russo and Vallois sense [18] when $H \in (1/4, 1/2)$ (resp. $H \in (1/2, 1)$). Also a method based on an extension of this Stratonovich integral is presented in [24].

The aim of this paper is to use the chaos expansion approach to show an existence and uniqueness result for linear stochastic differential equations of the form (1.1), in the case that $H \in (0, 1/2)$, the coefficients are deterministic and the stochastic integral is an extension of the divergence operator in the Malliavin calculus sense (see Proposition 2.5 below). The solution is given in terms of explicit expressions for the kernels of the fractional multiple integrals in its chaos expansion. Also a sufficient condition for the continuity of the solution is provided. This chaotic expansion procedure is introduced in Shiota [21] for $H = 1/2$. That is, when $B$ is a Brownian motion.

The organization of the article is as follows. Section 2 describes the framework of this paper. Namely, Section 2.1 introduces some basic elements of the fractional calculus and Section 2.2 gives some basic definitions and facts of the stochastic calculus for the fBm. In Section 3 we study equation (1.1). Finally, we develop the auxiliary tools needed for our proofs in Section 4.

# 2 Preliminaries

The purpose of this section is to describe the framework that will be used in this paper. Although some results discussed in this section are known, we prefer to provide a self–contained exposition for the convenience of the reader.

## 2.1 Fractional calculus

For a detailed account on the fractional calculus theory, we refer to Samko et al. [20].

Throughout $T$ is a positive number. Consider an integrable function $f : [0, T] \to \mathbb{R}$ and $\alpha \in (0, 1)$. The *right–sided fractional integral* of $f$ of order $\alpha$ is given by

$$I^{\alpha}_{T-}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^T \frac{f(u)}{(u-x)^{1-\alpha}} du, \quad \text{for a.a. } x \in [0, T].$$

Note that Fubini theorem implies that $I^{\alpha}_{T-}(f)$ is a function in $L^p([0, T]), p \geq 1$, whenever $f \in L^p([0, T])$. That is, the space $L^p([0, T])$ is invariant under the right–sided fractional integral $I^{\alpha}_{T-}$. Actually we have that $I^{\alpha}_{T-}$ has the following property (see [20], Theorem 3.5 and Notes §4):

**Lemma 2.1** *Let $1 < p < 1/\alpha$. Then $I^{\alpha}_{T-}$ is a linear bounded operator from $L^p([0, T])$ into $L^r([0, T])$, for every $1 \leq r \leq p(1 - \alpha p)^{-1}$.*

In what follows $C(p,r)$ denotes the norm of $I^{\alpha}_{T-}$ as a bounded linear operator from $L^p([0,T])$ into $L^r([0,T])$.

We denote by $I^{\alpha}_{T-}(L^p), p \geq 1$, the family of all functions $f \in L^p([0,T])$ such that

$$f = I^{\alpha}_{T-}(\varphi), \tag{2.1}$$

for some $\varphi \in L^p([0,T])$. Samko et al. [20] (Theorem 13.2) provide a characterization of the space $I^{\alpha}_{T-}(L^p)$, $p > 1$. Namely, a measurable function $f$ belongs to $I^{\alpha}_{T-}(L^p)$ (i.e., it satisfies (2.1)) if and only if $f \in L^p([0,T])$ and the integral

$$\int_{s+\varepsilon}^{T} \frac{f(s) - f(u)}{(u-s)^{1+\alpha}} du \tag{2.2}$$

converges in $L^p([0,T])$ as $\varepsilon \downarrow 0$. In this case a function $\varphi$ satisfying (2.1) agrees with the right–sided fractional derivative

$$(D^{\alpha}_{T-}f)(s) = \frac{1}{\Gamma(1-\alpha)} \Big( \frac{f(s)}{(T-s)^{\alpha}} + \alpha \int_s^T \frac{f(s) - f(u)}{(u-s)^{1+\alpha}} du \Big), \tag{2.3}$$

where the integral is the $L^p([0,T])$–limit of (2.2). Moreover, by [20] (Lemma 2.5), there is at most one solution $\varphi$ to the equation (2.1).

A useful tool to analyze the convergence of (2.2) is the following inequality.

**Lemma 2.2** *Let $0 < s < t < r$. Then*

$$\alpha \int_t^r (r-u)^{\alpha-1} u^{-\alpha} (u-s)^{-\alpha-1} du \leq t^{-\alpha} (t-s)^{-\alpha} (r-s)^{-1} (r-t)^{\alpha}.$$

*Proof:* The result is an immediate consequence of the changes of variables $z = (r-u)(u-s)^{-1}$ and $y = (t-s)z$, and the fact that $y \mapsto \frac{(t-s)r+ys}{y+t-s}$ is a decreasing function on $[0, r-t]$. ■

In the remaining of this paper, we will consider the space

$$\Lambda^{\alpha}_T := \{ f : [0,T] \to \mathbb{R} : \exists\, \phi_f \in L^2([0,T]), \text{ such that } f(u) = u^{\alpha} I^{\alpha}_{T-}(s^{-\alpha}\phi_f(s))(u) \}. \tag{2.4}$$

Here $\alpha \in (0, \frac{1}{2})$. More precisely, we will make use of the following result.

**Lemma 2.3** *Let $f \in \Lambda^{\alpha}_T$ be such that $\phi_f \in L^p([0,T])$ for some $p \in (2, 1/\alpha)$. Then $f 1_{[0,t]}$ also belongs to $\Lambda^{\alpha}_T$ for each $t \in [0,T]$, and for any $p' \in [2,p)$,*

$$||\phi_{f1_{[0,t]}}||_{L^{p'}([0,T])} \leq C_{\alpha,p',p,t} ||\phi_f||_{L^p([0,T])},$$

*where* $C_{\alpha,p',p,t} = t^{(p-p')/p'p} + \frac{C(p, p(1-\alpha p)^{-1})}{\Gamma(1-\alpha)} (\int_0^t (t-s)^{-p'\alpha q} ds)^{1/p'q}$ *with* $q = \frac{p}{p-p'(1-\alpha p)}$.

**Remarks.**

i) The assumptions of the result give that $1 - p'\alpha q > 0$.

ii) Remember that $C(p, p(1-\alpha p)^{-1})$ is the norm of the linear operator $I^{\alpha}_{T-} : L^p([0,T]) \to L^{p(1-\alpha p)^{-1}}([0,T])$ (see Lemma 2.1).

*Proof:* Fix $t \in [0,T]$. Then, by (2.3), we have

$$\begin{aligned}(D^{\alpha}_{T-}(u^{-\alpha}f(u)1_{[0,t]}(u)))(s) &= 1_{[0,t]}(s)\Big[s^{-\alpha}\phi_f(s) \\ &\quad +\frac{\alpha}{\Gamma(1-\alpha)}\int_t^T \frac{u^{-\alpha}f(u)}{(u-s)^{1+\alpha}}du\Big].\end{aligned}$$

That is (see (2.4)),

$$\phi_{f1_{[0,t]}}(s) = 1_{[0,t]}(s)\Big[\phi_f(s) + \frac{\alpha s^{\alpha}}{\Gamma(1-\alpha)}\int_t^T \frac{u^{-\alpha}f(u)}{(u-s)^{1+\alpha}}du\Big]. \tag{2.5}$$

Finally, observe that Lemma 2.2 gives

$$\begin{aligned}&||(\cdot)^{\alpha}1_{[0,t]}(\cdot)\int_t^T \frac{u^{-\alpha}|f(u)|}{(u-\cdot)^{1+\alpha}}du||_{L^{p'}([0,T])} \\ &\leq \frac{1}{\alpha}||1_{[0,t]}(\cdot)(t-\cdot)^{-\alpha}\frac{1}{\Gamma(\alpha)}\int_t^T \frac{|\phi_f(r)|}{(r-\cdot)^{1-\alpha}}dr||_{L^{p'}([0,T])}.\end{aligned}$$

Hence, Lemma 2.1 implies that for $q = \frac{p}{p-p'(1-\alpha p)}$,

$$\begin{aligned}&||(\cdot)^{\alpha}1_{[0,t]}(\cdot)\int_t^T \frac{u^{-\alpha}|f(u)|}{(u-\cdot)^{1+\alpha}}du||_{L^{p'}([0,T])} \\ &\leq \frac{C(p,p(1-\alpha p)^{-1})}{\alpha}(\int_0^t (t-s)^{-p'\alpha q}ds)^{1/p'q}||\phi_f||_{L^p([0,T])},\end{aligned}$$

which, together with (2.5), yields that the Lemma holds. ■

We will also need the following result.

**Lemma 2.4** *Let $f$ be a function in $\Lambda^{\alpha}_T$ and $g:[0,T]\to\mathbb{R}$ a Hölder continuous function with parameter $\beta>\alpha$. Then $gf$ also belongs to $\Lambda^{\alpha}_T$.*

*Proof:* Using equality (2.3) again, we obtain

$$\begin{aligned}(D^{\alpha}_{T-}(u^{-\alpha}g(u)f(u)))(s) &= g(s)(D^{\alpha}_{T-}(u^{-\alpha}f(u)))(s) \\ &\quad +\frac{\alpha}{\Gamma(1-\alpha)}\int_s^T u^{-\alpha}f(u)\frac{g(s)-g(u)}{(u-s)^{\alpha+1}}du.\end{aligned}$$

Finally, it follows from the Hölder continuity of $g$ that $s \mapsto s^{\alpha}\int_s^T u^{-\alpha}|f(u)|\frac{|g(s)-g(u)|}{(u-s)^{\alpha+1}}du$ is a square–integrable function. Thus $gf$ belongs to $\Lambda^{\alpha}_T$. ■

## 2.2 Fractional Brownian motion

Throughout $B^H = \{B^H_t : t \in [0,T]\}$ is a fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1/2)$ defined on a complete probability space $(\Omega,\mathcal{F},P)$. It means, the fBm $B^H$ is a Gaussian process with zero mean and covariance function

$$R_H(t,s) := \frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H}), \quad s,t\in[0,T].$$

In the remaining of this paper, we assume $\mathcal{F} = \sigma\{B_t : t \in [0,T]\}$. The reader can consult Nualart [15] and references therein for a recent presentation of the facts related to the fBm.

Let $\mathcal{H}_H$ be the Hilbert space defined as the completion of the step functions on $[0,T]$ with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}_H} = R_H(t,s) \quad t,s \in [0,T].$$

From Pipiras and Taqqu [17] (see also [15]), it follows that $\mathcal{H}_H$ coincides with the Hilbert space $\Lambda_T^{1/2-H}$ (introduced in (2.4)) equipped with scalar product

$$\langle f,g\rangle_{\Lambda_T^{1/2-H}} = C_H \langle \phi_f, \phi_g\rangle_{L^2([0,T])},$$

with $C_H = \frac{2H\Gamma(H+\frac{1}{2})}{(1-2H)\beta(1-2H,H+\frac{1}{2})}$. So the map $1_{[0,t]} \mapsto B_t^H$ is extended to an isometry of $\Lambda_T^{1/2-H}$ onto a Guassian closed subspace of $L^2(\Omega,\mathcal{F},P)$ (see Nualart [15]). This isometry is denoted by $\phi \mapsto B^H(\phi)$.

Now we assume that the reader is familiar with the basic elements of the stochastic calculus of variations for Gaussian processes as given for example in Nualart [14].

Let $n$ be a positive integer. The $n$-th multiple integral $I_n$ of order $n$ with respect to $B^H$ is a linear operator from the $n$–th symmetric tensor product $(\Lambda_T^{1/2-H})^{\odot n}$ of $\Lambda_T^{1/2-H}$ into $L^2(\Omega,\mathcal{F},P)$ satisfying the following two properties:

- Let $H_m$ be the $m$–th Hermite polynomial

$$H_m(x) = \frac{(-1)^m}{m!} e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2}, \quad x \in \mathbb{R},$$

  and $\{e_k : k \in \mathbb{N}\}$ an orthonormal system on $\Lambda_T^{1/2-H}$. Then, for any $f_n \in (\Lambda^{1/2-H})^{\odot n}$, we have

$$\begin{aligned} &E[I_n(f_n)(n_1!)H_{n_1}(B^H(e_{i_1}))\cdots(n_k)!H_{n_k}(B^H(e_{i_k}))] \\ &= \begin{cases} n!\langle f_n, e_{i_1}^{\otimes n_1}\otimes\cdots\otimes e_{i_k}^{\otimes n_k}\rangle_{(\Lambda_T^{1/2-H})^{\otimes n}}, & \text{if} \quad n=\sum_{j=1}^k n_j \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

- Let $f \in (\Lambda_T^{1/2-H})^{\odot m}$ and $g \in (\Lambda_T^{1/2-H})^{\odot n}$. Then

$$E[I_m(f)I_n(g)] = \begin{cases} 0, & \text{if} \quad n \neq m, \\ m!\langle f,g\rangle_{(\Lambda_T^{1/2-H})^{\otimes m}}, & \text{if} \quad n=m. \end{cases}$$

As a consequence of the relation between multiple integrals and Hermite polynomials, we have that any $F \in L^2(\Omega,\mathcal{F},P)$ has a unique chaotic representation of the form

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

with $I_0(f_0) = EF$.

León and Nualart [9] have extended the domain of the divergence operator in the sense of Malliavin calculus for Gaussian stochastic processes. This extension was first analyzed by Cheridito and Nualart [5] when the underlying Gaussian process is the fBm $B = \{B_t : t \in \mathbb{R}\}$ with Hurst parameter $H \in (0, 1/2)$. For the fBm $B^H$, this extension denoted by $\delta$ is characterized by the following result (see [9]).

**Proposition 2.5** *Let $u \in L^2(\Omega; L^2([0,T]))$ be a random variable with the chaos representation*

$$u = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in (\Lambda_T^{1/2-H})^{\odot n} \otimes L^2([0,T]).$$

*Then $u \in$ Dom $\delta$ iff $\widetilde{f}_n$ (the symmetrization of $f_n$ in $L^2([0,T]^{n+1})$) is in $(\Lambda_T^{1/2-H})^{\odot(n+1)}$ for every $n \in \mathbb{N}$ and*

$$\sum_{n=1}^{\infty} n! ||\widetilde{f}_{n-1}||_{(\Lambda_T^{1/2-H})^{\otimes n}} < \infty.$$

*In this case $\delta(u) = \sum_{n=1}^{\infty} I_n(\widetilde{f}_{n-1})$.*

**Remarks.**

i) The space $\Lambda_T^{\alpha}$ is included in $L^2([0,T])$ for any $\alpha \in (0, 1/2)$, by the Fubini's theorem.

ii) In [5] and [9], it is shown that the domain of $\delta$ is bigger than that of the usual divergence operator. Also Hu [6] (Section 7.1) has considered a set of integrable processes including Dom $\delta$. The reader can see Decreusefond and Üstünel [4] for a related construction of a stochastic integral with respect to $B^H$.

iii) In Section 3, we use the convention

$$\int_0^t u_s dB_s^H = \delta(u 1_{[0,t]}),$$

whenever $u 1_{[0,t]} \in$ Dom $\delta$.

# 3 Linear fractional differential equations

In the remaining of this paper $B^H$ is a fBm with Hurst parameter $H \in (0, 1/2)$ and we use the notation

$$\alpha = \frac{1}{2} - H.$$

In this section we study the chaos decomposition of the solution to a linear stochastic differential equation of the form

$$X_t = \eta + \int_0^t a(s) X_s ds + \int_0^t b(s) X_s dB_s^H, \quad t \in [0,T]. \tag{3.1}$$

Here $\eta$ is a square–integrable random variable having the chaotic representation

$$\eta = \sum_{n=0}^{\infty} I_n(\eta_n), \tag{3.2}$$

$a, b$ are two functions in $L^2([0,T])$ such that $b$ is also in $\Lambda_T^\alpha$.

## 3.1 Statement of problem and main results

Suppose that equation (3.1) has a solution $X$ in $L^2(\Omega \times [0,T])$ with the chaos decomposition

$$X_t = \sum_{n=0}^{\infty} I_n(f_n^t), \quad f_n \in (\Lambda_T^\alpha)^{\odot n} \otimes L^2([0,T]). \tag{3.3}$$

Then the uniqueness of the chaotic representation (3.3), and Proposition 2.5 imply

$$f_0^t = \eta_0 \exp(\int_0^t a(s)ds), \quad t \in [0,T],$$

and

$$\begin{aligned} f_n^t(t_1,\ldots,t_n) &= \eta_n(t_1,\ldots,t_n) + \int_0^t a(s) f_n^s(t_1,\ldots,t_n)ds \\ &\quad + \frac{1}{n}\sum_{j=1}^{n} b(t_j) f_{n-1}^{t_j}(t_1,\ldots,\hat{t}_j,\ldots,t_n) 1_{[0,t]}(t_j), \quad t,t_1,\ldots,t_n \in [0,T]. \end{aligned}$$

Hence, using induction on $n$, we obtain

$$\begin{aligned} &f_n^t(t_1,\ldots,t_n) = \exp(\int_0^t a(s)ds)\Big[\eta_n(t_1,\ldots,t_n) \\ &+ \sum_{j=1}^{n}\sum_{\Delta_{j,n}} \frac{(n-j)!}{j!n!} b^{\otimes j}(t_{i_1},\ldots,t_{i_j})\eta_{n-j}(\hat{t}_{i_1},\ldots,\hat{t}_{i_j}) 1_{[0,t]^j}(t_{i_1},\ldots,t_{i_j})\Big]. \end{aligned} \tag{3.4}$$

Here we use the convention

$$\Delta_{j,n} = \{\{i_1,\ldots,i_j\} \subset \{1,\ldots,n\} : i_k \neq i_\ell \text{ if } k \neq \ell\}.$$

Consequently, equation (3.1) has at most one solution in $L^2(\Omega \times [0,T])$.

Conversely, suppose that the functions $f_n$ given by (3.4) satisfy the following conditions:

1. For every $n \geq 0$, $f_n \in (\Lambda_T^\alpha)^{\odot n} \otimes L^2([0,T])$.

2. The process $Y_t = \sum_{n=0}^{\infty} I_n(f_n^t)$ belongs to $L^2(\Omega \times [0,T])$. That is,

$$\sum_{n=0}^{\infty} n! \int_0^T ||f_n^t||^2_{(\Lambda_T^\alpha)^{\otimes n}} dt < \infty. \tag{3.5}$$

3. For almost all $t \in [0,T]$, $bY1_{[0,t]}$ belongs to Dom $\delta$.

Then the process $Y$ is a solution in $L^2(\Omega \times [0,T])$ of equation (3.1).

The approach just described allows us to show the following result, which is one our goals in this paper.

Henceforth we use the notation

$$B_{H,p,t} = 1 + C_H \Big( B_{T,p} ||\phi_{b1_{[0,t]}}||_{L^p([0,T])} + ||\phi_{b1_{[0,t]}}||_{L^2([0,T])} \Big)^2$$

with

$$B_{T,p} = \frac{T^{(p-2)/2p}}{\Gamma(1-\alpha)} C(p, p/(1-\alpha p)) \left( \frac{p - 2(1-\alpha p)}{p-2} \right)^{(p-2(1-p\alpha))/2p}.$$

**Theorem 3.1** *Let $p \in (2, 1/\alpha)$. Assume that $\phi_b \in L^p([0,T])$ and*

$$\sum_{k=0}^{\infty} (k+1)! ||\eta_k||^2_{(\Lambda_T^\alpha)^{\otimes k}} (\sup_{t\in[0,T)} B_{H,\tilde{p},t})^k < \infty \tag{3.6}$$

*for some $\tilde{p} \in (2,p)$. Then equation (3.1) has a unique solution in $L^2(\Omega \times [0,T])$ given by*

$$X_t = \sum_{n=0}^{\infty} I_n(f_n^t),$$

*where $f_n$ is defined in (3.4).*

**Remarks 3.2** *i)* In Lemma 2.3 we have found a bound for $||\phi_{b1_{[0,t]}}||_{L^{\tilde{p}}([0,T])}$, $\tilde{p} \in [2,p)$.

*ii)* The following are examples of initial conditions that satisfy Hypothesis (3.6):

*a)* $\eta$ has a finite chaos decomposition. That is $\eta = \sum_{n=0}^{M} I_n(\eta_n)$, $M < \infty$.

*b)* $\eta$ has exponential growth. It means, there is a positive constant $C$ such that

$$||\eta_n||_{(\Lambda_T^\alpha)^{\otimes n}} \leq C^n (n!)^{-1}.$$

*c)* There is $\varepsilon > 0$ such that

$$\sum_{k=0}^{\infty} (k!)^{1+\varepsilon} ||\eta_k||^2_{(\Lambda_T^\alpha)^{\otimes n}} < \infty.$$

*iii)* By (3.4) we have that the solution $X$ of equation (3.1) has the form

$$X_t = \exp\Big(\int_0^t a(s)ds\Big) Y_t,$$

where $Y$ is the solution in $L^2(\Omega \times [0,T])$ to the equation

$$Y_t = \eta + \int_0^t b(s) Y_s dB_s, \quad t \in [0,T].$$

The Kolmogorov's continuity criterion implies the following result, which is our second goal in this paper.

**Theorem 3.3** *Let $p, \phi_b$ and $\eta$ be as in Theorem 3.1. Moreover assume*

$$\sum_{k=1}^{\infty} e^{k\theta} k^{k/2} ||\eta_k||_{(\Lambda_T^\alpha)^{\otimes k}} < \infty, \tag{3.7}$$

*for some $\theta$ such that $(1+e^{2\theta})(\frac{p-2}{2p} \wedge \alpha \wedge \frac{(1-\alpha p)}{p}) > 1$. Then the solution of equation (3.1) has a continuous version in $L^2(\Omega \times [0,T])$.*

**Remarks.**

i) Observe that (3.6) holds (resp. (3.7) is not true) when

$$||\eta_k||_{(\Lambda_T^\alpha)^{\otimes k}} = (k^2(k+1)!(\sup_{t\in[0,T)} B_{H,\tilde{p},t})^k)^{-1/2}$$

(resp. and $e^\theta \geq (\sup_{t\in[0,T)} B_{H,\tilde{p},t})$).

ii) Condition (3.7) does not necessarily imply Assumption (3.6) when $\theta$ is such that $e^{\theta+\varepsilon} \leq (\sup_{t\in[0,T)} B_{H,\tilde{p},t})^{1/2}$ for some $\varepsilon > 1/2$. Indeed, if we have

$$||\eta_k||_{(\Lambda_T^\alpha)^{\otimes k}} = (\sup_{t\in[0,T)} B_{H,\tilde{p},t})^{-k/2}((k+1)!)^{-1/2}.$$

Then(3.7) is satisfied but (3.6) does not hold.

iii) Remark 3.2 ii) is an example of initial conditions that satisfy (3.6) and (3.7) at the same time.

## 3.2 Proof of the main results

We begin this section with the proof of Theorem 3.1.

*Proof of Theorem 3.1:* We will follow the method (Steps 1–3) indicated in Section 3.1.

Let $f_n$ be given by (3.4). Then for each $t \in [0,T]$,

$$\begin{aligned} ||f_n^t||_{(\Lambda_T^\alpha)^{\otimes n}} &\leq \exp(\int_0^t a(s)ds)\Big[||\eta_n||_{(\Lambda_T^\alpha)^{\otimes n}} \\ &\qquad + \sum_{j=1}^{n}(j!)^{-1}||b1_{[0,t]}||^j_{\Lambda_T^\alpha}||\eta_{n-j}||_{(\Lambda_T^\alpha)^{\otimes(n-j)}}\Big] \\ &= \exp(\int_0^t a(s)ds)\sum_{j=0}^{n}\frac{||b1_{[0,t]}||^{n-j}_{\Lambda_T^\alpha}}{(n-j)!}||\eta_j||_{(\Lambda_T^\alpha)^{\otimes j}}. \end{aligned} \tag{3.8}$$

Thus Lemma 2.3 yields that $f_n \in (\Lambda_T^\alpha)^{\odot n} \otimes L^2([0,T])$.

Now we see that (3.5) is satisfied. By (3.8) we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} n! ||f_n^t||^2_{(\Lambda_T^\alpha)^{\otimes n}} \\
&\leq \exp(2\int_0^t a(s)ds) \sum_{n=0}^{\infty} n! \left(\sum_{j=0}^{n} \frac{||b1_{[0,t]}||^{2(n-j)}_{\Lambda_T^\alpha}}{j!(n-j)!}\right)\left(\sum_{j=0}^{n} j! \frac{||\eta_j||^2_{(\Lambda_T^\alpha)^{\otimes j}}}{(n-j)!}\right) \\
&= \exp(2\int_0^t a(s)ds) \sum_{n=0}^{\infty} (||b1_{[0,t]}||^2_{\Lambda_T^\alpha} + 1)^n \sum_{j=0}^{n} j! \frac{||\eta_j||^2_{(\Lambda_T^\alpha)^{\otimes j}}}{(n-j)!} \\
&= \exp(2\int_0^t a(s)ds + ||b1_{[0,t]}||^2_{\Lambda_T^\alpha} + 1) \sum_{j=0}^{\infty} j! ||\eta_j||^2_{(\Lambda_T^\alpha)^{\otimes j}} (||b1_{[0,t]}||^2_{\Lambda_T^\alpha} + 1)^j.
\end{aligned}
$$

Consequently, from (3.6) and Lemma 2.3 it follows that (3.5) holds and

$$
\sum_{n=0}^{\infty} n! \int_0^T b(t)^2 ||f_n^t||^2_{(\Lambda_T^\alpha)^{\otimes n}} dt < \infty. \tag{3.9}
$$

Set $Y_t = \sum_{n=0}^{\infty} I_n(f_n^t)$. So, to finish the proof, we only need to show that for each $t \in [0,T]$, the process $bY1_{[0,t]}$ belongs to Dom $\delta$.

By Lemmas 4.5, 4.6 and 4.7 below we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} (n+1)! || \frac{1}{n+1} \sum_{k=1}^{n+1} 1_{[0,t]}(t_k) b(t_k) f_n^{t_k}(\hat{t}_k) ||^2_{(\Lambda_T^\alpha)^{\otimes(n+1)}} \leq \\
&2\exp\left(2\int_0^T |a(s)|ds + 1 + C_H ||\phi_{b1_{[0,t]}}||^2_{L^2([0,T])}\right) \sum_{k=0}^{\infty} k! ||\eta_k||^2_{(\Lambda_T^\alpha)^{\otimes k}} (1 + C_H ||\phi_{b1_{[0,t]}}||^2_{L^2([0,T])})^k \\
&+2\Big(\frac{A\alpha}{\Gamma(1-\alpha)}\Big)^2 C_H \exp(B_{H,\tilde{p},t}) \Big\{\sum_{k=0}^{\infty} (k+1)! ||\eta_k||^2_{(\Lambda_T^\alpha)^{\otimes k}} (B_{H,\tilde{p},t})^k + \sum_{k=0}^{\infty} k! ||\eta_k||^2_{(\Lambda_t^\alpha)^{\otimes k}} (B_{H,\tilde{p},t})^{k+1}\Big\}.
\end{aligned}
$$

Thus the result follows from (3.6), (3.9) and Proposition 2.5. ∎

Now we give the proof of Theorem 3.3.

*Proof of Theorem 3.3:* We first note that we can assume that $a = 0$ using Remark 3.2. iii). So in this case, by the Hypercontractivity property (see [14]) and (3.4), the solution $X$ of equation (3.1) satisfies for $q = 1 + e^{2\theta}$,

$$
\begin{aligned}
E|X_t - X_s|^q &\leq [\sum_{n=1}^{\infty} ||I_n(f_n^t - f_n^s)||_{L^q(\Omega)}]^q \\
&\leq (t-s)^{\delta q} [\sum_{n=1}^{\infty} e^{n\theta} \sqrt{n!} \sum_{j=1}^{n} \frac{1}{j!} ||\eta_{n-j}||_{(\Lambda_T^\alpha)^{\otimes(n-j)}} (C||\phi_b||_{L^p([0,T])})^j]^q, \qquad (3.10)
\end{aligned}
$$

where the last inequality follows from Corollary 4.9 below.

Now observe that there exists a positive constant $\tilde{C}$ such that

$$\begin{aligned}
&\sum_{n=1}^{\infty} e^{n\theta}\sqrt{n!}\sum_{j=1}^{n}\frac{1}{j!}||\eta_{n-j}||_{(\Lambda_T^\alpha)^{\otimes(n-j)}}(C||\phi_b||_{L^p([0,T])})^j \\
&= \sum_{k=0}^{\infty}||\eta_k||_{(\Lambda_T^\alpha)^{\otimes k}}\sum_{n=k+1}^{\infty}\frac{\sqrt{n!}e^{n\theta}}{(n-k)!}(C||\phi_b||_{L^p([0,T])})^{n-k} \\
&\leq \sum_{k=0}^{\infty} e^{k\theta}||\eta_k||_{(\Lambda_T^\alpha)^{\otimes k}}\sum_{n=1}^{\infty}\sqrt{(n+k)^k}\frac{(e^\theta C||\phi_b||_{L^p([0,T])})^n}{\sqrt{n!}} \\
&\leq \sum_{k=0}^{\infty} e^{k\theta}||\eta_k||_{(\Lambda_T^\alpha)^{\otimes k}}k^{k/2}\sum_{n=1}^{\infty}\frac{(e^{\theta+\frac{1}{2}}C||\phi_b||_{L^p([0,T])})^n}{\sqrt{n!}} \\
&\leq \tilde{C}\sum_{k=0}^{\infty} e^{k\theta}k^{k/2}||\eta_k||_{(\Lambda_T^\alpha)^{\otimes k}}. \qquad (3.11)
\end{aligned}$$

Finally, the Kolmogorov's continuity theorem, together with (3.10) and (3.11), completes the proof. ■

# 4 Appendix

In this section the basic tool for our proofs is established.

Let $x \in \mathbb{R}^n$ and $(r_{i_1}, \ldots r_{i_j}) \in \mathbb{R}^j$, $j \leq n$. Henceforth we use the notation

$$\begin{aligned}
\Delta_k^x(r_{i_1},\ldots,r_{i_j}) \;=\; \Big\{ & u \in \mathbb{R}^n : u_i = x_i \text{ for } i \notin \{i_1,\ldots,i_j\}, \\
& u_i \in \{r_i, x_i\} \text{ for } i \in \{i_1,\ldots,i_j\} \text{ and} \\
& \#\{u_{i_1},\ldots,u_{i_j}\} \cap \{r_{i_1},\ldots,r_{i_j}\} = k \Big\},
\end{aligned}$$

$S_{j,n}$ for the class of ordered subsets

$$\{i_1 < \cdots < i_j\} \subset \Delta_{j,n} = \{\{i_1,\ldots,i_j\} \subset \{1,\ldots,n\} : i_k \neq i_\ell \text{ if } k \neq \ell\},$$

and $e(t) = \exp(\int_0^t a(s)ds)$.

**Lemma 4.1** *Let $f \in \Lambda_T^\alpha$. Then for every $n \in \mathbb{N}$, $(t_1,\ldots,t_n) \mapsto f^{\otimes n}(t_1,\ldots,t_n)e(t_1 \vee \cdots \vee t_n) \in (\Lambda_T^\alpha)^{\otimes n}$ and*

$$\begin{aligned}
&f^{\otimes n}(t_1,\ldots,t_n)e(t_1\vee\cdots\vee t_n)(t_1\cdots t_n)^{-\alpha} \\
&= \Gamma(\alpha)^{-n}\Big\{\int_{t_1}^{T}\cdots\int_{t_n}^{T}\frac{(\prod_{i=1}^{n}\phi_f(s_i)s_i^{-\alpha})e(s_1\vee\cdots\vee s_n)}{\prod_{i=1}^{n}(s_i-t_i)^{1-\alpha}}ds_n\cdots ds_1 \\
&\quad +\sum_{j=1}^{n}(\Gamma(1-\alpha))^{-j}\alpha^j\sum_{S_{j,n}}\int_{t_1}^{T}\cdots\int_{t_n}^{T}\frac{\prod_{i\notin\{i_1,\ldots,i_j\}}\phi_f(s_i)(s_i)^{-\alpha}}{\prod_{i=1}^{n}(s_i-t_i)^{1-\alpha}}
\end{aligned}$$

$$\cdot\Big[\int_{s_{i_1}}^T\cdots\int_{s_{i_j}}^T \frac{\prod_{k=1}^j f(u_{i_k})(u_{i_k})^{-\alpha}}{\prod_{k=1}^j (u_{i_k}-s_{i_k})^{1+\alpha}} \sum_{k=0}^j (-1)^k \sum_{\delta\in\Delta_k^s(u_{i_1},\dots,u_{i_j})} e(\delta_1\vee\cdots\vee\delta_n)du_{i_j}\cdots du_{i_1}\Big]ds_n\cdots ds_1\Big\}.$$

**Remark.** In Lemma 4.1 we are using the convention $\prod_{i\notin S_{n,n}}\phi_f(s_i)(s_i)^{-\alpha}=1$.

*Proof:* The result holds for $n=1$ due to Lemma 2.4. Now we use induction on $n$. So we assume that the result is true for $n$. Then, Lemma 2.4 implies

$$\begin{aligned}
& f(t_{n+1})t_{n+1}^{-\alpha}((\prod_{i=1}^n f(t_i)t_i^{-\alpha})e(t_1\vee\cdots\vee t_{n+1})) \\
= \; & (\Gamma(\alpha))^{-(n+1)}\Big\{\int_{t_1}^T\cdots\int_{t_{n+1}}^T \frac{\prod_{i=1}^n \phi_f(s_i)s_i^{-\alpha}}{\prod_{i=1}^{n+1}(s_i-t_i)^{1-\alpha}}\Big[e(s_1\vee\cdots\vee s_{n+1})\phi_f(s_{n+1})s_{n+1}^{-\alpha} \\
& +\frac{\alpha}{\Gamma(1-\alpha)}\int_{s_{n+1}}^T \phi_f(u)u^{-\alpha}\frac{e(s_1\vee\cdots\vee s_{n+1})-e(s_1\vee\cdots\vee s_n\vee u)}{(u-s_{n+1})^{1+\alpha}}du\Big]ds_{n+1}\cdots ds_1\Big\} \\
& +(\Gamma(\alpha))^{-n}\sum_{j=1}^n\left(\frac{\alpha}{\Gamma(1-\alpha)}\right)^j\sum_{S_{j,n}}\int_{t_1}^T\cdots\int_{t_n}^T\frac{\prod_{i\notin\{i_1,\dots,i_j,n+1\}}\phi_f(s_i)s_i^{-\alpha}}{\prod_{i=1}^n(s_i-t_i)^{1-\alpha}} \\
& \cdot\int_{s_{i_1}}^T\cdots\int_{s_{i_j}}^T\frac{\prod_{k=1}^j f(u_{i_k})u_{i_k}^{-\alpha}}{\prod_{k=1}^j(u_{i_k}-s_{i_k})^{1+\alpha}}\sum_{k=0}^j(-1)^k\sum_{\delta\in\Delta_k^{(s_1,\dots,s_n)}(u_{i_1},\dots,u_{i_j})} \\
& \cdot\Big[\frac{1}{\Gamma(\alpha)}\int_{t_{n+1}}^T\frac{\phi_f(s_{n+1})s_{n+1}^{-\alpha}e(\delta_1\vee\cdots\vee\delta_n\vee s_{n+1})}{(s_{n+1}-t_{n+1})^{1-\alpha}}ds_{n+1} \\
& +\frac{\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{t_{n+1}}^T(s_{n+1}-t_{n+1})^{\alpha-1}\int_{s_{n+1}}^T f(u_{n+1})(u_{n+1})^{-\alpha} \\
& \cdot\frac{e(\delta_1\vee\cdots\vee\delta_n\vee s_{n+1})-e(\delta_1\vee\cdots\vee\delta_n\vee u_{n+1})}{(u_{n+1}-s_{n+1})^{1+\alpha}}du_{n+1}ds_{n+1}\Big]du_{i_j}\cdots du_{i_1}ds_n\cdots ds_1 \\
= \; & \frac{1}{\Gamma(\alpha)^{n+1}}\int_{t_1}^T\cdots\int_{t_{n+1}}^T\frac{\prod_{i=1}^n\phi_f(s_i)s_i^{-\alpha}}{\prod_{i=1}^{n+1}(s_i-t_i)^{1-\alpha}}\Big[e(s_1\vee\cdots\vee s_{n+1})\phi_f(s_{n+1})s_{n+1}^{-\alpha} \\
& +\frac{\alpha}{\Gamma(1-\alpha)}\int_{s_{n+1}}^T\phi_f(u_{n+1})u_{n+1}^{-\alpha}\frac{e(s_1\vee\cdots\vee s_{n+1})-e(s_1\vee\cdots\vee s_n\vee u_{n+1})}{(u_{n+1}-s_{n+1})^{1+\alpha}}du_{n+1}\Big] \\
& \cdot ds_{n+1}\cdots ds_1 \\
& +\frac{1}{\Gamma(\alpha)^{1+n}}\sum_{j=1}^n(\frac{\alpha}{\Gamma(1-\alpha)})^j\sum_{S_{j,n}}\int_{t_1}^T\cdots\int_{t_{n+1}}^T\frac{(\prod_{i\notin\{i_1,\dots,i_j,n+1\}}\phi_f(s_i)(s_i^{-\alpha}))\phi_f(s_{n+1})s_{n+1}^{-\alpha}}{\prod_{i=1}^{n+1}(s_i-t_i)^{1+\alpha}}
\end{aligned}$$

$$\cdot \int_{s_{i_1}}^T \cdots \int_{s_{i_j}}^T \frac{\prod_{k=1}^j f(u_{i_k}) u_{i_k}^{-\alpha}}{\prod_{k=1}^j (u_{i_k} - s_{i_k})^{1+\alpha}} \sum_{k=0}^j (-1)^k \sum_{\delta \in \Delta_k^{\widehat{s_{n+1}}}(u_{i_1},\ldots,u_{i_j})} e(\delta_1 \vee \cdots \vee \delta_n \vee s_{n+1})$$

$$\cdot du_{i_j} \cdots du_{i_1} ds_{n+1} \cdots ds_1$$

$$+ \frac{1}{\Gamma(\alpha)^{1+n}} \sum_{j=2}^{n+1} (\frac{\alpha}{\Gamma(1-\alpha)})^j \sum_{S_{j-1,n}} \int_{t_1}^T \cdots \int_{t_{n+1}}^T \frac{\prod_{i \notin \{i_1,\ldots i_{j-1},n+1\}} \phi_f(u_i) u_i^{-\alpha}}{\prod_{i=1}^{n+1} (s_i - t_i)^{1-\alpha}}$$

$$\cdot \int_{s_{i_1}}^T \cdots \int_{s_{i_{j-1}}}^T \int_{s_{n+1}}^T \frac{(\prod_{k=1}^{j-1} f(u_{i_k}) u_{i_k}^{-\alpha}) f(u_{n+1}) u_{n+1}^{-\alpha}}{(\prod_{k=1}^{j-1} (u_{i_k} - s_{i_k})^{1+\alpha})(u_{n+1} - s_{n+1})^{1+\alpha}} \sum_{k=0}^j (-1)^k$$

$$\cdot \sum_{\delta \in \Delta_k^s(u_{i_1},\ldots,u_{i_{j-1}},u_{n+1})} e(\delta_1 \vee \cdots \vee \delta_{n+1}) du_{n+1} du_{i_{j-1}} \cdots du_{u_{i_1}} ds_{n+1} \cdots ds_1,$$

which proves that the result holds for $n+1$ whenever it is true for $n$. ∎

**Lemma 4.2** *Let $s \in [0,T]^n$, $n \geq 2$, $(i_1,\ldots,i_j) \in S_{j,n}$ and $(u_{i_1},\ldots,u_{i_j}) \in [s_{i_1},T] \times \cdots \times [s_{i_j},T]$. Then*

$$\begin{aligned} & \sum_{k=0}^j (-1)^k \sum_{\delta \in \Delta_k^s(u_{i_1},\ldots,u_{i_j})} e(\delta_1 \vee \cdots \vee \delta_n) \\ & \quad = \quad (e(s_1 \vee \cdots \vee s_n) - e(u_{i_1} \wedge \cdots \wedge u_{i_j})) 1_{[s_1 \vee \cdots \vee s_n, T]^j}(u_{i_1},\ldots,u_{i_j}). \end{aligned}$$

*Proof:* The proof follows from the fact that

$$\begin{aligned} & \sum_{k=0}^{j+1} (-1)^k \sum_{\delta \in \Delta_k^s(u_{i_1},\ldots,u_{i_j})} e(\delta_1 \vee \cdots \vee \delta_n) \\ & = \sum_{k=0}^j (-1)^k \sum_{\delta \in \Delta_k^{\widehat{(s_{i_{j_0}})}}(\widehat{u_{i_{j_0}}})} [e(\delta_1 \vee \cdots \vee \delta_{n-1} \vee s_{i_{j_0}}) - e(\delta_1 \vee \cdots \vee \delta_{n-1} \vee u_{i_{j_0}})], \end{aligned}$$

and induction on $j$. ∎

**Lemma 4.3** *Let $f \in \Lambda_T^\alpha$ be such that $\phi_f \in L^p([0,T])$ for some $p \in (2, 1/\alpha)$. Then*

$$\begin{aligned} & || \prod_{i \notin \{i_1,\ldots,i_j\}} \phi_f(s_i) \int_{[s_1 \vee \cdots \vee s_n, T]^j} \\ & \quad \cdot \frac{(\prod_{k=1}^j f(u_{i_k}) u_{i_k}^{-\alpha} s_{i_k}^\alpha)(e(s_1 \vee \cdots \vee s_n) - e(u_{i_1} \wedge \cdots \wedge u_{i_j}))}{\prod_{k=1}^j (u_{i_k} - s_{i_k})^{1+\alpha}} du_{i_1} \cdots du_{i_j} ||^2_{L^2([0,T]^n)} \\ & \leq \quad j A^2 B_p^{2(j-1)} ||\phi_f||^{2(j-1)}_{L^p([0,T])} ||\phi_f||^{2(n+1-j)}_{L^2([0,T])}, \end{aligned} \tag{4.1}$$

*with*

$$A = \exp\left(\int_0^T |a(s)|ds\right) ||a||_{L^2([0,T])} \left(\int_0^1 \frac{d\theta}{(1-\theta)^{1-\alpha}\theta^{\frac{1}{2}+\alpha}}\right) \frac{2T}{\Gamma(\alpha)}$$

*and*

$$B_p = \frac{T^{(p-2)/2p}}{\alpha} \left(\frac{p-2(1-\alpha p)}{p-2}\right)^{\frac{p-2(1-\alpha p)}{2p}} C(p, p/(1-\alpha p)).$$

*Proof:* Lemma 2.2 implies that the left–hand side of (4.1) is bounded by

$$\begin{aligned}
& j! ||\phi_f||^{2(n-j)}_{L^2([0,T])} \exp(2\int_0^T |a(s)|ds) ||a||^2_{L^2([0,T])} \\
& \quad \cdot \int_0^T \int_0^{s_j} \cdots \int_0^{s_2} \left((s_1 \ldots s_j)^\alpha \int_{[s_j,T]^j} \frac{\prod_{i=1}^j |f(u_i)| u_i^{-\alpha}}{\prod_{i=1}^j (u_i - s_i)^{1+\alpha}} |u_j - s_j|^{1/2} du_1 \ldots du_j\right)^2 ds_1 \ldots ds_j \\
\leq \; & \frac{j}{\alpha^{2(j-1)}} \exp\left(2\int_0^T |a(s)|ds\right) ||a||^2_{L^2([0,T])} \left(\int_0^1 \frac{(1-\theta)^{\alpha-1}}{\theta^{\frac{1}{2}+\alpha}} d\theta\right)^2 ||\phi_f||^{2(n-j)}_{L^2([0,T])} \\
& \quad \cdot \int_0^T \left(\frac{1}{\Gamma(\alpha)} \int_{s_j}^T \frac{|\phi_f(u)|}{(u-s_j)^{1/2}} du\right)^2 \left[\int_0^{s_j} (s_j - s)^{-2\alpha} \left(\frac{1}{\Gamma(\alpha)} \int_{s_j}^T \frac{|\phi_f(u)|}{(u-s)^{1-\alpha}} du\right)^2 ds\right]^{j-1} ds_j.
\end{aligned}$$

Hence, it is quite easy to finish the proof using Lemma 2.1. ■

**Lemma 4.4** *Let $f$ be as in Lemma 4.3. Then*

$$\begin{aligned}
& ||f^{\otimes n}(t_1, \ldots, t_n) e(t_1 \vee \ldots \vee t_n)||_{(\Lambda_T^\alpha)^{\otimes n}} \\
\leq \; & C_H^{n/2} \Big\{ ||\phi_f||^n_{L^2([0,T])} \exp(\int_0^T |a(s)|ds) + \\
& \frac{nA\alpha}{\Gamma(1-\alpha)} \left(\frac{\alpha}{\Gamma(1-\alpha)} B_p ||\phi_f||_{L^p([0,T])} + ||\phi_f||_{L^2([0,T])}\right)^{n-1} \Big\}.
\end{aligned}$$

*Proof:* The proof is an immediate consequence of Lemmas 4.1, 4.2 and 4.3. ■

**Lemma 4.5** *Let $f_n$ be given by (3.4). Moreover assume that $\phi_b \in L^p([0,T])$, $p \in (2, 1/\alpha)$. Then*

$$\begin{aligned}
& \sum_{n=0}^\infty (n+1)! || \frac{1}{n+1} \sum_{k=1}^{n+1} 1_{[0,t]}(t_k) b(t_k) f_n^{t_k}(\hat{t}_k) ||^2_{(\Lambda_T^\alpha)^{\otimes(n+1)}} \\
\leq \; & 2\exp\left(2\int_0^t |a(s)|ds\right) \sum_{n=0}^\infty (n+1)! \left[\sum_{k=0}^n \frac{1}{(n+1-k)!} ||\eta_k||_{(\Lambda_T^\alpha)^{\otimes k}} C_H^{(n+1-k)/2} ||\phi_{b1_{[0,t]}}||^{n+1-k}_{L^2([0,T])}\right]^2 \\
& +2\left(\frac{\alpha}{\Gamma(1-\alpha)}\right)^2 A^2 \sum_{n=0}^\infty (n+1)! \left[\sum_{k=0}^n \frac{1}{(n-k)!} C_H^{(n+1-k)/2} ||\eta_k||_{(\Lambda_T^\alpha)^{\otimes k}} \right. \\
& \quad \left. \cdot \left(\frac{\alpha}{\Gamma(1-\alpha)} B_{\tilde{p}} ||\phi_{b1_{[0,t]}}||_{L^{\tilde{p}}([0,T])} + ||\phi_{b1_{[0,t]}}||_{L^2([0,T])}\right)^{n-k}\right]^2,
\end{aligned}$$

*where $\tilde{p} \in (2, p)$ and $A, B_{\tilde{p}}$ are given in Lemma 4.3.*

*Proof:* The definition of $f_n$ gives

$$\begin{aligned}
&\frac{1}{n+1}\sum_{k=1}^{n+1} 1_{[0,t]}(t_k) f_n^{t_k}(\hat{t}_k) b(t_k) \\
&= \sum_{k=1}^{n+1} \frac{1}{n+1} b(t_k) \exp(\int_0^{t_k} a(s)ds) \eta_n(\hat{t}_k) 1_{[0,t]}(t_k) \\
&+ \sum_{j=1}^{n} \sum_{\Delta_{j+1,n+1}} \frac{(n-j)!}{(n+1)!} b^{\otimes(j+1)}(t_{i_1}, \ldots, t_{i_{j+1}}) \eta_{n-j}(\hat{t}_{i_1}, \ldots \hat{t}_{i_{j+1}}) \\
&\quad \cdot \exp(\int_0^{t_{i_{j+1}}} a(s)ds) 1_{\{t_{t_1} < \cdots < t_{j+1} < t\}} \\
&= \sum_{k=1}^{n+1} \sum_{\Delta_{k,n+1}} \frac{(n+1-k)!}{(n+1)!k!} (\prod_{j=1}^{k} b(t_{i_j}) 1_{[0,t]}(t_{i_j})) \eta_{n+1-k}(\hat{t}_{i_1}, \ldots, \hat{t}_{i_k}) e(t_{i_1} \vee \cdots \vee t_{i_k}).
\end{aligned}$$

Hence Lemma 4.4 gives

$$\begin{aligned}
&\sum_{n=0}^{\infty} (n+1)! || \frac{1}{n+1} \sum_{k=1}^{n+1} 1_{[0,t]}(t_k) b(t_k) f_n^{t_k}(\hat{t}_k) ||^2_{(\Lambda_T^\alpha)^{\otimes(n+1)}} \\
&\leq \sum_{n=0}^{\infty} (n+1)! \left[ \sum_{k=1}^{n+1} \frac{1}{k!} ||\eta_{n+1-k}||_{(\Lambda_T^\alpha)^{\otimes(n+1-k)}} C_H^{k/2} \Big\{ ||\phi_{b1_{[0,t]}}||^k_{L^2([0,T])} \exp(\int_0^t |a(s)|ds) \right. \\
&\quad \left. + \frac{Ak\alpha}{\Gamma(1-\alpha)} \left( \frac{\alpha}{\Gamma(1-\alpha)} B_{\tilde{p}} ||\phi_{b1_{[0,t]}}||_{L^{\tilde{p}}([0,T])} + ||\phi_{b1_{[0,T]}}||_{L^2([0,T])} \right)^{k-1} \Big\} \right]^2,
\end{aligned}$$

and the result follows. ∎

**Lemma 4.6** *Assume that $\phi_b \in L^p([0,T])$ for some $p \in (2, 1/\alpha)$. Then*

$$\begin{aligned}
&\sum_{n=0}^{\infty} (n+1)! \left[ \sum_{k=0}^{n} \frac{1}{(n+1-k)!} ||\eta_k||_{(\Lambda_T^\alpha)^{\otimes k}} C_H^{(n+1-k)/2} ||\phi_{b1_{[0,t]}}||^{n+1-k}_{L^2([0,T])} \right]^2 \leq \\
&\exp\left(1 + C_H ||\phi_{b1_{[0,t]}}||^2_{L^2([0,T])}\right) \sum_{k=0}^{\infty} k! ||\eta_k||^2_{(\Lambda_T^\alpha)^{\otimes k}} (1 + C_H ||\phi_{b1_{[0,t]}}||^2_{L^2([0,T])})^k.
\end{aligned} \tag{4.2}$$

*Proof:* Observe that the left–hand side of (4.2) is bounded by

$$\begin{aligned}
&\sum_{n=0}^{\infty} (n+1)! \Big[ \sum_{k=0}^{n} \frac{k!}{(n+1-k)!} ||\eta_k||^2_{(\Lambda_T^\alpha)^{\otimes k}} \Big] \Big[ \sum_{k=0}^{n} \frac{(C_H)^{n+1-k}}{k!(n+1-k)!} ||\phi_{b1_{[0,t]}}||^{2(n+1-k)}_{L^2([0,T])} \Big] \\
&\leq \sum_{k=0}^{\infty} k! ||\eta_k||^2_{(\Lambda_T^\alpha)^{\otimes k}} (1 + C_H ||\phi_{b1_{[0,t])}}||^2_{L^2([0,T])})^k \sum_{n=k}^{\infty} \frac{(1 + C_H ||\phi_{b1_{[0,T]}}||^2_{L^2([0,T])})^{n+1-k}}{(n+1-k)!}.
\end{aligned}$$

Thus, the proof is complete. ∎

**Lemma 4.7** *Let $\tilde{p} \in (2, p)$. Then*

$$\sum_{n=0}^{\infty}(n+1)!\Big[\sum_{k=0}^{n}\tfrac{C_H^{(n+1-k)/2}}{(n-k)!}||\eta_k||_{(\Lambda_T^\alpha)^{\otimes k}}\Big(\tfrac{\alpha}{\Gamma(1-\alpha)}B_{\tilde{p}}||\phi_{b1_{[0,t]}}||_{L^{\tilde{p}}([0,T])}+||\phi_{b1_{[0,t]}}||_{L^2([0,T])}\Big)^{n-k}\Big]^2$$
$$\leq C_H\exp(B_{H,\tilde{p},t})\Big\{\sum_{k=0}^{\infty}(k+1)!||\eta_k||^2_{(\Lambda_T^\alpha)^{\otimes k}}(B_{H,\tilde{p},t})^k+\sum_{k=0}^{\infty}k!||\eta_k||^2_{(\Lambda_T^\alpha)^{\otimes k}}(B_{H,\tilde{p},t})^{k+1}\Big\}, \quad (4.3)$$

*where*

$$B_{H,\tilde{p},t} = 1 + C_H\left(\frac{\alpha}{\Gamma(1-\alpha)}B_{\tilde{p}}||\phi_{b1_{[0,t]}}||_{L^{\tilde{p}}([0,T])}+||\phi_{b1_{[0,t]}}||_{L^2([0,T])}\right)^2.$$

*Proof:* Note that the left–hand side of (4.3) is dominated by

$$C_H\sum_{n=0}^{\infty}(n+1)\sum_{k=0}^{n}\frac{k!}{(n-k)!}||\eta_k||^2_{(\Lambda_T^\alpha)^{\otimes k}}(B_{H,\tilde{p},t})^n =$$
$$C_H\sum_{k=0}^{\infty}k!||\eta_k||^2_{(\Lambda_T^\alpha)^{\otimes k}}(B_{H,\tilde{p},t})^k\sum_{n=k}^{\infty}\frac{n+1}{(n-k)!}(B_{H,\tilde{p},t})^{n-k} \leq$$
$$C_H\exp(B_{H,\tilde{p},t})\sum_{k=0}^{\infty}k!||\eta_k||^2_{(\Lambda_T^\alpha)^{\otimes k}}(B_{H,\tilde{p},t})^k\Big(1+k+B_{H,\tilde{p},t}\Big).$$

Consequently, the result holds. ∎

We will use the following results in the proof of Theorem 3.3.

**Lemma 4.8** *Let $p \in (2, 1/\alpha)$ and $\phi_b \in L^p([0,T])$. Then for any $\delta < (\frac{p-2}{2p}\wedge\alpha\wedge\frac{(1-\alpha p)}{p})$ there is a positive constant $C_\delta$ such that*

$$\begin{aligned}&||(b1_{[0,t]})^{\otimes n}-(b1_{[0,s]})^{\otimes n}||_{(\Lambda_T^\alpha)^{\otimes n}}\\ \leq\ & 2^{n-1}(C_\delta)^n||\phi_b||^n_{L^p([0,T])}(t-s)^\delta, \quad 0\leq s\leq t\leq T.\end{aligned}$$

*Proof:* We will use induction on $n$ to prove the result.

First assume that $n = 1$. In this case, by (2.5), we have

$$\begin{aligned}&||b1_{[0,t]}-b1_{[0,s]}||_{\Lambda_T^\alpha}\\ \leq\ & C_H^{1/2}\Big\{||\phi_b 1_{]s,t]}||_{L^2([0,T])}+\frac{\alpha}{\Gamma(1-\alpha)}\Big(\int_0^s\Big(r^\alpha\int_s^t\frac{b(u)u^{-\alpha}}{(u-r)^{1+\alpha}}du\Big)^2dr\Big)^{1/2}\\ &+\frac{\alpha}{\Gamma(1-\alpha)}\Big(\int_s^t\Big(r^\alpha\int_t^T\frac{b(u)u^{-\alpha}}{(u-r)^{1+\alpha}}du\Big)^2dr\Big)^{1/2}\Big\}\\ =\ & C_H^{1/2}\Big\{I_1+\frac{\alpha}{\Gamma(1-\alpha)}I_2+\frac{\alpha}{\Gamma(1-\alpha)}I_3\Big\}.\end{aligned} \quad (4.4)$$

It is clear that we have

$$I_1 \leq (t-s)^{(p-2)/2p}||\phi_b||_{L^p([0,T])}. \quad (4.5)$$

Now, from Lemmas 2.1 and 2.2, it follows

$$
\begin{aligned}
I_3 &\le \frac{1}{\Gamma(\alpha)\alpha}\Big[\Big(\int_s^t (t-r)^{-2\alpha}\Big(\int_r^T \frac{|\phi_b(u)|}{(u-r)^{1-\alpha}}du\Big)^2 dr\Big]^{1/2} \\
&\le \frac{C(p,p/(1-\alpha p))}{\alpha\Gamma(\alpha)}\left(\frac{p-2(1-\alpha p)}{p-2}\right)^{(p-2(1-\alpha p))/2p} ||\phi_b||_{L^p([0,T])}(t-s)^{(p-2)/2p}. \quad (4.6)
\end{aligned}
$$

On the other hand, the Fubini's theorem and Lemma 2.2 lead to

$$
\begin{aligned}
&1_{[0,s]}(r)r^\alpha\int_s^t \frac{|b(u)u^{-\alpha}|}{(u-r)^{1+\alpha}}du \\
&\le 1_{[0,s]}(r)\frac{r^\alpha}{\Gamma(\alpha)}\int_s^T |\phi_b(\theta)|\theta^{-\alpha}(\int_s^{t\wedge\theta}\frac{du}{(u-r)^{1+\alpha}(\theta-u)^{1-\alpha}})d\theta \\
&= 1_{[0,s]}(r)\frac{r^\alpha}{\Gamma(\alpha)}\Big[\int_s^t |\phi_b(\theta)|\theta^{-\alpha}(\int_s^{\theta}\frac{du}{(\theta-u)^{1-\alpha}(u-r)^{1+\alpha}})d\theta \\
&\qquad + \int_t^T |\phi_b(\theta)|\theta^{-\alpha}(\int_s^{t}\frac{du}{(u-r)^{1+\alpha}(\theta-u)^{1-\alpha}})d\theta\Big] \\
&\le 1_{[0,s]}(r)\Big\{\frac{(t-s)^\delta}{\Gamma(\alpha)\alpha}(s-r)^{-\alpha}\int_s^t \frac{|\phi_b(\theta)|}{(\theta-r)^{1-(\alpha-\delta)}}d\theta \\
&\qquad + \frac{1}{\Gamma(\alpha)}(s-r)^{-\alpha-\delta}\int_s^t \frac{1}{(u-r)^{1-\delta}}(\int_t^T \frac{|\phi_b(\theta)|}{(\theta-u)^{1-\alpha}}d\theta)du\Big\}. \quad (4.7)
\end{aligned}
$$

Therefore, combining (4.4) and (4.7), we get

$$
\begin{aligned}
I_2 &\le \frac{(t-s)^\delta}{\Gamma(\alpha)\alpha}(\int_0^s (s-r)^{-2\alpha}(\int_r^T \frac{|\phi_b(\theta)|}{(\theta-r)^{1-(\alpha-\delta)}}d\theta)^2 dr)^{1/2} \\
&\quad + \frac{1}{\Gamma(\alpha)}\Big(\int_0^s (s-r)^{-2(\alpha+\delta)}\Big[\int_s^t \frac{1}{(u-r)^{1-\delta}}(\int_u^T \frac{|\phi_b(\theta)|}{(\theta-u)^{1-\alpha}}d\theta)du\Big]^2 dr\Big)^{1/2} \\
&= \frac{(t-s)^\delta}{\Gamma(\alpha)\alpha}I_{2,1} + \frac{1}{\Gamma(\alpha)}I_{2,2}. \quad (4.8)
\end{aligned}
$$

Observe that Lemma 2.1 gives

$$
I_{2,1} \le [C(p,\frac{p}{1-p(\alpha-\delta)})(\frac{T^{1-2q\alpha}}{1-2q\alpha})^{1/q}]^{1/2}||\phi_b||_{L^p([0,T])}, \quad (4.9)
$$

with $q = p(p-2(1-p(\alpha-\delta)))^{-1}$. Similarly, we have for $\delta < \frac{p-2}{p} \wedge \frac{2(1-\alpha p)}{p}$,

$$
\begin{aligned}
I_{2,2} &\le \Big[\int_0^s (s-r)^{-2(\alpha+\delta)}(\int_s^t \frac{du}{(u-r)^{1-\delta}}) \\
&\qquad \cdot(\int_s^t \frac{1}{(u-r)^{1-\delta}}(\int_u^T \frac{|\phi_b(\theta)|}{(\theta-u)^{1-\alpha}}d\theta)^2 du)dr\Big]^{1/2} \\
&\le \frac{(t-s)^{\delta/2}}{\sqrt{\delta}}\Big[\int_0^s (s-r)^{-2(\alpha+\delta)}\int_r^T \frac{1}{(u-r)^{1-\delta}}(\int_u^T \frac{|\phi_b(\theta)|}{(\theta-u)^{1-\alpha}}d\theta)^2 du dr\Big]^{1/2} \\
&\le \frac{(t-s)^{\delta/2}}{\sqrt{\delta}}[\frac{T^{1-2(\alpha+\delta)\tilde{p}}}{1-2(\alpha+\delta)\tilde{p}}]^{1/2\tilde{p}}C(p,\frac{p}{1-\alpha p})C(\frac{p}{2(1-\alpha p)},\frac{p}{2(1-\alpha p)-\delta p})||\phi_b||_{L^p([0,T])},
\end{aligned}
$$

with $\tilde{p} = p(p-2(1-\alpha p)+\delta p)^{-1}$. Consequently, from (4.4), (4.5), (4.6),(4.8) and (4.9), we have that the results holds for $n=1$.

Finally we can apply induction on $n$ to show the result is true for any $n$, due to Lemma 2.3 and the fact that

$$\begin{aligned} &||(b1_{[0,t]})^{\otimes n} - (b1_{[0,s]})^{\otimes n}||_{(\Lambda_T^\alpha)^{\otimes n}} \\ &\leq \ ||b1_{[0,s]}||_{\Lambda_T^\alpha}||(b1_{[0,t]})^{\otimes(n-1)} - (b1_{[0,s]})^{\otimes(n-1)}||_{(\Lambda_T^\alpha)^{\otimes(n-1)}} \\ &\quad +||b1_{]s,t]}||_{\Lambda_T^\alpha}||(b1_{[0,t]})^{\otimes(n-1)}||_{(\Lambda_T^\alpha)^{\otimes(n-1)}}. \end{aligned}$$

■

An immediate consequence of Lemma 4.8 is the following.

**Corollary 4.9** *Let $p, \delta$ and $\phi_b$ as in Lemma 4.8. Then there is a constant $C$ such that*

$$\begin{aligned} &||\sum_{j=1}^{n}\sum_{\Delta_{j,n}}\frac{(n-j)!}{n!j!}\eta_{n-j}(\hat{t}_{i_1},\ldots,\hat{t}_{i_j})\Big(\prod_{k=1}^{j} b(t_{i_k})1_{[0,t]}(t_{i_k}) - \prod_{k=1}^{j} b(t_{i_k})1_{[0,s]}(t_{i_k})\Big)||_{(\Lambda_T^\alpha)^{\otimes n}} \\ &\qquad \leq \ (t-s)^\delta \sum_{j=1}^{n}\frac{1}{j!}||\eta_{n-j}||_{(\Lambda_T^\alpha)^{\otimes n-j}} C^j ||\phi_b||^j_{L^p([0,T])}. \end{aligned}$$

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# References

[1] E. Alòs, J. A. León and D. Nualart, Stochastic Stratonovich calculus for fractional Brownian motion with Hurst parameter less than 1/2, *Taiwanesse J. Math.* **5** (2001), 609–632.

[2] L. Coutin and Z. Qian, Stochastic differential equations for fractional Brownian motions, *C.R. Acad. Sci. Paris Sér. I Math.* **331** (2000), 75–80.

[3] L. Coutin and Z. Qian, Stochastic analysis, rough paths analysis and fractional Brownian motions, *Probab. Theory Related Fields* **122** (2002), 108–140.

[4] L. Decreusefond and A. S. Üstünel, Stochastic analysis of the fractional Brownian motion, *Potential Analysis* **10** (1999), 177–214.

[5] P. Cheridito and D. Nualart, Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$, Preprint.

[6] Y. Hu, Integral transformations and anticipative calculus for fractional Brownian motions, Preprint.

[7] M. L. Kleptsyna, P. E. Kloeden and V. V. Ahn, Existence and uniqueness theorems for fBm stochastic differential equations, *Problems of Inform. Transmission* **34** (1998), 332–341.

[8] K. Kubilius, The existence and uniqueness of the solution of the integral equation driven by fractional Brownian motion, *Liet. Mat. Rink.* **40** (2000), 104–110.

[9] J. A. León and D. Nualart, An extension of the divergence operator for Gaussian processes, *Stochastic Processes and Their Applications*. To appear.

[10] J. A. León and C. Tudor, Semilinear fractional stochastic differential equations, *Bol. Soc. Mat. Mexicana* **8** (2002), 205–226.

[11] S. J. Lin, Stochastic analysis of fractional Brownian motions, *Stochastics Stochastics Rep.* **55** (1995), 121–140.

[12] T. Lyons, Differential equations driven by rough signals, *Rev. Mat. Iberoamericana* **14** (1998), 215–310.

[13] B. B. Mandelbrot and J. W. Van Ness, Fractional Brownian motions, fractional noises and applications, *SIAM Rev.* **10** (1968), 422–437.

[14] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer–Verlag, 1995.

[15] D. Nualart, Stochastic integration with respect to fractional Brownian motion and applications, In: *Stochastic Models, Proceedings of the Seventh Symposium on Probability and Stochastic Processes*, ed. by J. M. González–Barrios et. al. Contemporary Mathematics **336** (2003), 3–39.

[16] D. Nualart and A. Răşcanu, Differential equations driven by fractional Brownian motion, *Collect. Math.* **53** (2002), 55–81.

[17] V. Pipiras and M. S. Taqqu, Are classes of deterministic integrands for fractional Brownian motion on an interval complete?, *Bernoulli* **7** (2001), 873–897.

[18] F. Russo and P. Vallois, Forward, backward and symmetric stochastic integration, *Probab. Theory Related Fields* **97** (1993), 403–421.

[19] A. A. Ruzmaikina, Stieltjes integrals of Hölder continuous functions with applications to fractional Brownian motion, *J. Statist. Phys.* **100** (2000), 1049–1069.

[20] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, 1987.

[21] Y. Shiota, A linear stochastic integral equation containing the extended Itô integral. *Math. Rep. Toyana Univ.* **9** (1986), 43–65.

[22] L. C. Young, An inequality of the Hölder type connected with Stieltjes integration, *Acta Math.* **67** (1936), 251–282.

[23] M. Zähle, Integration with respect to fractal functions and stochastic calculus I, *Probab. Theory Related Fields* **111** (1998), 333–374.

[24] M. Zähle, Integration with respect to fractal functions and stochastic calculus II, *Math. Nachr.* **225** (2001), 145–183.

[25] M. Zähle, On the link between fractional and stochastic calculus, In: *Stochastic Dynamics*, Bremen 1997, ed: H. Cravel and M. Gundlach, 305–325, Springer, 1999.